Recall:

what does $y = \log_b x$ mean?

It means that $b^y = x$, i.e., $\log_b x$ is the exponent which you need to raise the base b to get x

Properties of Logarithms

1. $\log_b 1 = 0$

why? $b^? = 0$

well $b^0 = 1$. So $\log_b 1 = 0$

2. $\log_b b = 1$

why? $b^? = b$

Since $b^1 = b$, $\log_b b = 1$

3. $\log_b b^x = x$

why? $b^? = b^x$

$? = x$. So, $\log_b b^x = x$

4. $b^{\log_b x} = x$

why? $\log_b x$ is the exponent which when raised with respect to b gives us x. Thus,

$$b^{\log_b x} = x$$

5> $\log_b MN = \log_b M + \log_b N$ (PRODUCT RULE)

Why?

Let $\log_b M = x$ and $\log_b N = y$. Then

$b^x = M$ and $b^y = N$.

We have $b^x \cdot b^y = MN$

$\Rightarrow b^{x+y} = MN$

$\therefore \log_b MN = x+y = \log_b M + \log_b N$

6. $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$ (QUOTIENT RULE)

Why? Let $x = \log_b M$ and $y = \log_b N$. Then
$b^x = M$ and $b^y = N$.

we have, $\frac{b^x}{b^y} = \frac{M}{N}$

or, $b^{x-y} = \frac{M}{N}$

$\therefore \log_b\left(\frac{M}{N}\right) = x - y = \log_b M - \log_b N.$

7. $\log_b M^p = p \log_b M$ (POWER RULE)

Why? Let $X = \log_b M$. Then
$b^x = M$.

we have, $(b^x)^p = M^p$ (raising both sides to p)
$b^{px} = M^p$

$\therefore \log_b M^p = px$
$= p \log_b M$

Basic properties of common and natural logarithms

Common log (base 10)

1. $\log 1 = 0$
2. $\log 10 = 1$
3. $\log 10^{x} = x$
4. $10^{\log x} = x \quad x > 0$

Natual log. (Base e)

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln e^{x} = x$
4. $e^{\ln x} = x \quad x > 0$

Exercise

Simplify the following:

(a) $\log_{10} 10$
$= 1$

(b) $\ln 1$
$= 0$

(c) $10^{\log(x+8)}$
$= x + 8 \qquad (x > -8)$

(d) $e^{\ln(2x+5)}$
$= 2x + 5 \qquad (x > -5/2)$

(e) $\log 10^{x^2}$
$= x^2$

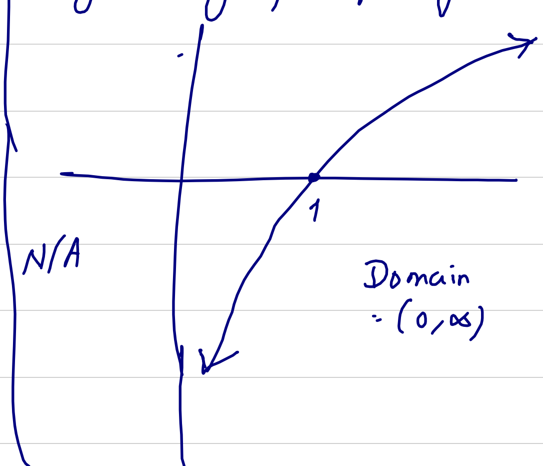
(f) $\ln e^{x+3}$
$= x + 3$

Note

Domain of $\log_b$ is always $(0, \infty)$.

Why? Because you cannot a negative number or zero if you raise b to any exponent.

This can also be seen by the graph of $\log_b$



Mnemonic

log turns product to sum. $\left(\log(MN) = \log M + \log N \right.$

log turns division to difference $\left. \log\left(\frac{M}{N}\right) = \log M - \log N \right)$

Exercise

Simplify

$$\log_b(u^2\sqrt{v})$$
$$= \log_b(u^2) + \log_b(\sqrt{v})$$
$$= \log_b(u^2) + \log_b(v^{1/2})$$
$$= 2\log_b(u) + \frac{1}{2}\log_b(v)$$

(Power rule for log $\log_b M^p = p\log_b M$)

Exercise

Simplify $\log_b(x^4\sqrt[3]{y})$

Exercise

Write $2\ln x + 3\ln y$ as a single logarithm.

Exercise

Write $\ln\left(\frac{x^3}{y^2}\right)$ as diff. of log.

Soln $\ln\left(\frac{x^3}{y^2}\right) = \ln(x^3) - \ln(y^2)$ (Quotient rule)

$= 3\ln x - 2\ln y$ (Power rule.)

Exercise

write $\log\left(\frac{a^4}{b^5}\right)$ as difference of log.

Exercise

write $\frac{2}{3}\ln x - \frac{1}{2}\ln y$ as a logarithm of quotient

Soln. $\frac{2}{3}\ln x - \frac{1}{2}\ln y$

$= \ln x^{2/3} - \ln y^{1/2}$ (Power rule)

$= \ln\left(\frac{x^{2/3}}{y^{1/2}}\right)$

Exercise

write $\frac{1}{2}\log a - 3\log b$ as single logarithm.

Exercise

write $3\log_b x + \log_b(2x+1) - 2\log_b 4$ as single log.

Soln. $3\log_b x + \log_b(2x+1) - 2\log_b 4$

$= \log_b x^3 + \log_b(2x+1) - \log_b 4^2$

$= \log_b\left(x^3(2x+1)\right) - \log_b 16$

$= \log_b\left[\frac{x^3(2x+1)}{16}\right]$

Exercise write $2\ln x - \ln(3y) + 3\ln z$ as single log.

Change of Base Formula

Example: Assume your calculator can calculate only commmm logarithms, i.e. base 10. Then evaluate $\log_3 8$.

Ans. Let $x = \log_3 8$

Then, $3^x = 8$

Taking log on both sides we get (log stands for base 10)

$$\log 3^x = \log 8$$

$$\Rightarrow x \log 3 = \log 8 \quad \text{(Power Rule)}$$

$$\Rightarrow x = \frac{\log 8}{\log 3}$$

$$\therefore \log_3 8 = \frac{\log 8}{\log 3}$$

In general, if you know $\log_a M$ and want to calculate the $\log_b M$ (with base b) the formula is

$$\log_b M = \frac{\log_a M}{\log_a b}$$

Bonus Exercise Prove it

Hint: Same as the previous calculation.

We will only change to base 10 or base e.
So we need to know two formulas

COMMON

$$\log_b M = \frac{\log M}{\log b}$$

NATURAL

$$\log_b M = \frac{\ln M}{\ln b}$$

Exercise

Use change of base formula to evaluate $\log_4 17$.
Use both common and natural log.

Soln.

By change of base formula,

$$\log_4 17 = \frac{\log 17}{\log 4}$$

$$\approx 2.0437$$

Also,

$$\log_4 17 = \frac{\ln 17}{\ln 4}$$

$$\approx 2.0437$$

Exercise

Use change of base formula to approximate $\log_7 34$.